Supplemental Notes

EE503 Week 07 Dr. Franzke

HW #7

1. Leon-Garcia 5.8, 5.12–5.14

   5.25 – 5.26, 5.35, 5.74

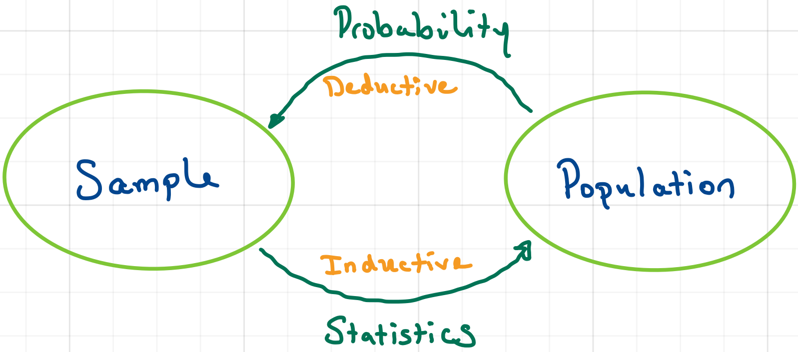
Topics

1. Multiple random variables
2. Uncertainty Principle

$$\sigma_{XY}^2 \leq \sigma_X^2 \sigma_Y^2$$

3. Sample vs. Population

Sample vs. Population



<u>Data</u>

$x_1, x_2, x_3, \ldots$

<u>Assumptions</u>

$X \sim N(\mu, \sigma^2)$

Statistic = f (Data)

<u>Sample</u> | <u>Population</u>

$\bar{X}_n \xrightarrow{\text{LLN}} \mu_X \quad (E_X[X])$

$S_X^2 \xrightarrow{\text{LLN}} \sigma_X^2$

Latin r.v.

$S_X \qquad \sigma_X$

Greek constant

$S_{XY} \qquad \sigma_{XY}$

$R_{XY} \qquad \rho_{XY}$

$S_{XY}^2 \overset{\text{u.p.}}{\leq} S_X^2 S_Y^2 \qquad \sigma_{XY}^2 \overset{\text{u.p.}}{\leq} \sigma_X^2 \sigma_Y^2$

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k$$

$$S_x^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2$$

$$S_{xy} = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \overline{X}_n)(Y_k - \overline{Y}_n)$$

matrix: $\underset{p \times m}{\hat{K}_{xy}} = \frac{1}{n-1} \sum_{k=1}^{n} \underset{p \times 1}{(X_k - \overline{X}_n)} \underset{1 \times m}{(Y_k - \overline{Y}_n)^T}$

Defn: The <u>covariance</u> of X and Y:

$$\sigma_{XY} \triangleq \mathrm{Cov}(X, Y)$$

$$= \underbrace{E_{XY}}_{\text{for joint pdf } f_{XY}(x,y)}\left[(X-\mu_x)(Y-\mu_y)\right] \quad \text{(if exists)}$$

∴ **Facts:**

① $\sigma_{XY} = E_{XY}[XY] - E_X[X] \cdot E_Y[Y]$.

② $E_{XY}[aX + bY] = a \cdot E_X[X] + b \cdot E_Y[Y]$

③ $V_{XY}[aX + bY] = a^2 \cdot V_X[X] + b^2 V_Y[Y] + 2ab\,\sigma_{XY}$

④ $V\left[\sum_{k=1}^{n} c_k \cdot X_k\right] = \sum_{k=1}^{n} c_k^2 \sigma_{X_k}^2 + \sum_{k=1}^{n} \sum_{\substack{l=1 \\ k \neq l}}^{n} c_k c_l \sigma_{X_k X_l}$

$$= \sum_{k=1}^{n} c_k^2 \sigma_{X_k}^2 \quad \text{if } X_1 \ldots X_n \text{ independent}$$

⑤ $V[XY] = V[X] \cdot V[Y] + V[X] \cdot E^2[Y]$

$+ V[Y] \cdot E^2[X]$ if X and Y independent

Autocovariance: $\underset{n\times n}{K_{xx}} = E[\underset{n\times 1}{(X-\mu_x)}\underset{n\times 1}{(X-\mu_x)}^T]$

$= E[XX^T] - \mu_x \mu_x^T$

$\therefore$ K positive semi-definite since $\forall x \neq \vec{0}$

$x^T K x = E[(x^T(X-\mu_x))^2] \geq 0$

$\therefore$ K diagonable $\quad K = O^T \Lambda O$

Cross covariance $\quad \underset{n\times p}{K_{xy}} = E[\underset{n\times 1}{(X-\mu_x)}\underset{1\times p}{(Y-\mu_y)^T}]$.

$\therefore$ If $Z = \left(\frac{X}{Y}\right)$

$$K_{zz} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} K_{xx} & 0 \\ 0 & K_{yy} \end{pmatrix}$$ if X and Y uncorrelated

Thm: (Cauchy Schwartz Uncertainy Principle – U.P.) $\sigma_{XY}^2 \le \sigma_X^2 \sigma_Y^2$

Prf: $\forall c \in \mathbb{R}$

$$0 \le E_{XY}\left[\left((X-\mu_X) - c\cdot(Y-\mu_Y)\right)^2\right]$$

$$= E_{XY}\left[(X-\mu_X)^2 + c^2\cdot(Y-\mu_Y)^2 - 2c\cdot(X-\mu_X)(Y-\mu_Y)\right].$$

$$= E_{XY}\left[(X-\mu_X)^2\right] + c^2\cdot E_{XY}\left[(Y-\mu_Y)^2\right] - 2c\cdot E_{XY}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= E_X\left[(X-\mu_X)^2\right] + c^2\cdot E_Y\left[(Y-\mu_Y)^2\right] - 2c\sigma_{XY}$$

$$= \sigma_X^2 + c^2\cdot\sigma_Y^2 - 2c\cdot\sigma_{XY}$$

$\therefore$ put $c = \dfrac{\sigma_{XY}}{\sigma_Y^2}$ since holds $\forall c \in \mathbb{R}$.

$$= \sigma_X^2 + \frac{\sigma_{XY}^2}{\sigma_Y^4}\cdot\sigma_Y^2 - 2\cdot\frac{\sigma_{XY}}{\sigma_Y^2}\cdot\sigma_{XY}$$

$$= \sigma_X^2 + \frac{\sigma_{XY}^2}{\sigma_Y^2} - 2\cdot\frac{\sigma_{XY}^2}{\sigma_Y^2}$$

$$= \sigma_X^2 - \frac{\sigma_{XY}^2}{\sigma_Y^2}$$

$\therefore \dfrac{\sigma_{XY}^2}{\sigma_Y^2} \le \sigma_X^2$ $\qquad \therefore \sigma_{XY}^2 \le \sigma_X^2\sigma_Y^2$

$$\therefore \quad |\sigma_{XY}| \overset{\text{u.p.}}{\leq} \sigma_X \sigma_Y$$

$$\therefore \quad -\sigma_X \sigma_Y \leq \sigma_{XY} \leq \sigma_X \sigma_Y$$

$$\therefore \quad -1 \leq \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \leq 1$$

Defn: The population correlation coefficient:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\therefore \quad -1 \leq \rho_{XY} \leq 1$$

Defn: X and Y are Jointly Normal (Gaussian) iff

"JG" "Bivariate normal"

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \cdot \exp\left[-\frac{\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho_{XY}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]}{2(1-\rho_{XY}^2)}\right]$$

special case $\rho_{XY}=0$

$$= \frac{1}{2\pi\sigma_X\sigma_Y} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}\right)\left(\frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}\right)$$

$$= f_X(x) \cdot f_Y(y) \quad \text{and} \quad X \sim N(\mu_X, \sigma_X^2) \quad Y \sim N(\mu_Y, \sigma_Y^2)$$

$\therefore$ If X and Y are JG and uncorrelated then X and Y are independent

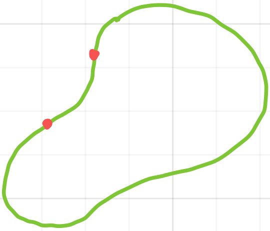
(false in general!)

Defn: f is convex iff

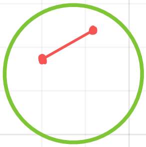
$$f(\lambda x + (1-\lambda) y) \leq \lambda \cdot f(x) + (1-\lambda) f(y)$$

for $\forall \lambda : 0 \leq \lambda \leq 1$

note: $f'' \geq 0 \rightarrow$ f convex.



Not Convex



Convex

(contains all points on any line segment)

$f(x) = e^x \implies f'' > 0$ $\therefore$ e^x is (strictly) convex

Thrm: (Jensen's Inequality) If f is convex:

$$f(E_x[X]) \leq E_x[f(X)]$$

(note: concave $\geq$)

mnemonic: "fief"

Ex: $f(x) = e^x$ convex

$$\therefore e^{E_x[X]} \leq E_x[e^x]$$

Jensen's Inequality $\Longrightarrow$ Generalized U.P.

☆ **Thrm:** (Hölder's Inequality) If $p > 1$ and $q > 2$

(∴ UP if $p = q = 2$)

and $\frac{1}{p} + \frac{1}{q} = 1$ then

$$E[|XY|] \le E^{1/p}[|X|^p] \cdot E^{1/q}[|Y|^q]$$

Thrm: (Lyapunov Inequality)

$$E^{1/\alpha}[|X|^\alpha] \le E^{1/\beta}[|X|^\beta] \quad \text{if } 0 < \alpha < \beta < \infty$$

Thrm: (Minkowski Inequality)

$$E_{XY}^{1/p}[|X+Y|^p] \le E_X^{1/p}[|X|^p] + E_Y^{1/p}[|Y|^p].$$

if $1 \le p < \infty$

Matrix Uncertainty Principle

scalar: $\sigma_{XY}^2 \le \sigma_X^2 \sigma_Y^2$

$\longleftrightarrow \quad \sigma_X^2 \ge \sigma_{XY}^2 (\sigma_Y^2)^{-1}$

$\longleftrightarrow \quad \sigma_X^2 - \sigma_{XY} (\sigma_Y^2)^{-1} \sigma_{YX} \ge 0.$

matrix: $K_{XX} - K_{XY} K_{YY}^{-1} K_{YX}$

is positive semi-definite ($w^T A w \ge 0 \ \forall w \ne \vec{0}$

iff $\forall$ eigenvalues $\lambda_k \ge 0$)

Recall: Matrix $\underset{n \times n}{A} \in \mathbb{C}^{n \times n}$ is diagonable

$A = O^T \Lambda O$ for $O^+ O = I = O O^+$ and

diagonal Λ (eigenvalues of A)

☆ iff L.I E.

$\longleftrightarrow$ A has n Linearly Independent Eigenvectors

$e_k: \ A e_k = \lambda_k e_k$

Always holds for real symmetric matrix:

Example: K_{XX}

Uncertainty Principle Applications

① Cramer-Rao Inequality

$$V[\hat{\theta}_n] \cdot J_n(\theta) \geq 1 \quad \text{if} \quad E[\hat{\theta}_n] = \theta \ \forall n.$$

② Conjugate Operators in Physics

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

③ Signal-processing Time-bandwidth products

$$V[\text{time signal}] \cdot V[\text{DTFT}] \geq \frac{1}{(4\pi)^2}$$

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① Cramer - Rao U.P.

- Estimator (r.v.) $\hat{\theta}_n$ estimates parameter θ
- "Unbiased" $E[\hat{\theta}_n] = \theta \ \forall n.$

Ex: $\hat{\theta}_n = \overline{X}_n$ (i.i.d.) for $\mu = \theta$

Fisher "Information" J:

$$J = E_{f(x|\theta)}\left[\left(\frac{\partial}{\partial \theta} \ln f(x|\theta)\right)^2\right] = E[S^2].$$

for iid sample $X_1, X_2, \ldots, X_n \sim f(x, \theta)$:

$$J_n(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\ln f(x_1,\dots,x_n\mid\theta)\right)^2\right].$$

$$\overset{iid}{=} \sum_{k=1}^{n} E\left[\left(\frac{\partial}{\partial\theta}\ln f(x_k\mid\theta)\right)^2\right].$$

$$\overset{iid}{=} n\cdot J$$

$$= V\Big[\underbrace{\frac{\partial}{\partial\theta}\ln f(x\mid\theta)}_{\text{score}}\Big]$$ since $E[\text{score}]=0$

$\therefore$ Fact: $\sigma_{\hat\theta_n} = Cov(\hat\theta_n, S) = 1$

$$\therefore\ V[\hat\theta_n]\cdot V[S] \geq 1$$

$$\therefore\ V[\hat\theta_n] \geq \frac{1}{J_n}$$ C.R. bound on variance of estimator $\hat\theta_n$

"=" holds $\longleftrightarrow$ $\hat\theta_n$ is "efficient" for θ.

Ex: iid $X_1,\dots,X_n \sim N(\theta, \sigma^2)$ ← known

$\therefore\ \overline{X}_n$ is efficient for μ

$$\left(J_n = \frac{1}{V[\overline{X}_n]} = \frac{n}{\sigma^2}\right)$$

② U.P. in Quantum Mechanics

★ Quantized energy $E = h\nu$ ← frequency (light photon)

h = Planck's constant $\left(\hbar = \frac{h}{2\pi}\right)$

$= 6.6 \times 10^{-29}$ erg·sec

Complex matter wave $\psi(x,t)$

" ∴ " ★ Schrodinger Wave Equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$ → time independent version 1-D

(reaction-diffusion equation)

$$= \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}}_{\text{kinetic energy part}} - \underbrace{V(x)\psi}_{\text{potential energy part}}$$

Stationary ("time independent" case) $\psi(x)$:

$$\hat{H}\psi = E\psi$$

∴ ψ is an eigenfunction

E is an eigenvalue

Energy

★ $|\psi|^2 = \psi^*\psi$

$=$ Probability (particle in dx)

$$\therefore \int_{-\infty}^{\infty} \psi^*\psi \, dx = 1$$

Assumption: "Observable" $\longleftrightarrow$ Hermitian operator A.

Hilbert space model

$A = A^{*T}$ if A matrix

: Only observe eigenvalues (energy) E:

$$E[A] = \int_{\mathbb{R}} \psi^* A \psi \, dx$$

$$= \int_{\mathbb{R}} \psi^* (E\psi) \, dx$$

since A Hermitian $A\psi = E\psi$.

$$= E \underbrace{\int_{\mathbb{R}} \psi^* \psi \, dx}_{=1}$$

$=1$ since $\psi^*\psi = |\psi|^2$ is a probability

$$= E$$

energy level or "spectrum"

Fact: If A and B are diagonable (and Hermitian) then A and B simultaneously diagonable

$$Ae = \lambda e \quad \text{and} \quad Be = \mu e$$

iff A and B commute: $AB = BA$

iff $AB - BA = 0$

Idea: Quantum U.P. $\longleftrightarrow$ two operators $AB \neq BA$.

"Commutator" notation: $[A,B] = AB - BA$

Position operator $\hat{x}$: multiply by x $(\hat{x} \cdot x)$

Newtonian momentum: $p = mv = m \cdot \frac{\partial}{\partial t} x$

Quantum momentum operator: $\hat{p}$ ("analogy")

$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ for $i = \sqrt{-1}$

$$\therefore [\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \left[x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right]$$

$$= \frac{\hbar}{i}\left[x, \frac{\partial}{\partial x}\right]$$

$$= -\frac{\hbar}{i}\left[\frac{\partial}{\partial x}, x\right]$$ since $[A,B] = AB - BA$.

$$= i \cdot \hbar \left[\frac{\partial}{\partial x}, x\right]$$ multiply $\frac{i}{i}$

$$= i\hbar\left(\frac{\partial}{\partial x} x - x \frac{\partial 1}{\partial x}\right)$$

$$= i\hbar(1 - 0)$$

$$= i \cdot \hbar$$

$$\neq 0$$

$\therefore$ $\hat{x}$ and $\hat{p}$ do <u>not</u> commute

Physics notation:

$$(\Delta A)^2 \triangleq V[A] = \sigma_A^2$$
$$= E[(A - E[A])^2]$$
$$= \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle$$ physics notation
$$= \int_{\mathbb{R}} \psi^* (\hat{A} - E[\hat{A}])^2 \psi \, dx$$

$\therefore$ Cauchy-Schwartz (and quadratic equation) gives:

☆ Fact: $$\Delta A \, \Delta B \geq \frac{1}{2} \left| \int_{\mathbb{R}} \psi^* [A, B] \psi \, dx \right|$$

Physical review 1929
H.P. Robertson

or

$$V[A] \cdot V[B] \geq \frac{1}{2} \left| E[(AB - BA)] \right|$$

$$\therefore \Delta x \, \Delta p \geq \frac{1}{2} \left| \int_{\mathbb{R}} \psi^* [\hat{x}, \hat{p}] \psi \, dx \right|$$
$$= \frac{1}{2} \left| \int_{\mathbb{R}} \psi^* (i\hbar) \psi \, dx \right|$$
$$= \frac{1}{2} \hbar \underbrace{|i|}_{=1} \cdot \underbrace{\int_{\mathbb{R}} \psi^* \psi \, dx}_{=1}$$
$$= \hbar/2$$

☆ $\therefore \quad \Delta x \Delta p \geq \frac{\hbar}{2}$ "the" uncertainty principle

or: $\sigma_x \sigma_p \geq \hbar/2$

Similarly $\Delta E \Delta t \geq \frac{\hbar}{2}$.

though time t is a parameter and not an operator

$\therefore$ zero energy is a precise value

$\therefore$ U.P. "forbids" it

$\therefore$ "Excommunication of the vacuum"

- vacuum has finite energy fluctuations

Q: source of nuclear decay?

③ Signal-Processing Uncertainty Principle ☆

time-signal $x(t)$ assume $\sum_{t=-\infty}^{\infty} |x(t)|^2 < \infty$

Define $\tilde{x}(t) = \dfrac{x(t)}{\sqrt{\sum_{t=-\infty}^{\infty} |x(t)|^2}}$ (normalize)

$$X(\omega) \overset{DTFT}{=} \sum_{t=-\infty}^{\infty} x(t) e^{-i\omega t}$$

Similarly define

$$\tilde{X}(\omega) = \frac{X(\omega)}{\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega}}$$

$\therefore$ Parseval: $\sum_{t=-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

$$\therefore \sum_{t=-\infty}^{\infty} |\tilde{x}(t)|^2 = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{X}(\omega)|^2 d\omega$$

$\therefore$ $|\tilde{x}(t)|^2$ and $\dfrac{|\tilde{X}(\omega)|^2}{2\pi}$ define pdfs

$\therefore$ In the time domain

$$\mu = \sum_{t=-\infty}^{\infty} t \cdot |\tilde{x}(t)|^2 \overset{\text{Assume}}{=} 0$$

$$\sigma^2 = \sum_{t=-\infty}^{\infty} (t-\mu)^2 |\tilde{x}(t)|^2 \overset{\mu=0}{=} \sum_{t=-\infty}^{\infty} t^2 \cdot |\tilde{x}(t)|^2 = V[\tilde{x}(t)]$$

In the frequency domain

Assume

$$\mu_{freq} = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} w \cdot |\tilde{X}(w)|^2 \, dw = 0$$

$$\sigma_{freq}^2 = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} (w - 2\pi \mu_{freq})^2 |\tilde{X}(w)|^2 \, dw$$

$\mu_{freq} = 0$

$$= \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} w^2 \cdot |\tilde{X}(w)|^2 \, dw$$

$$= V[\tilde{X}(w)]$$

Further assume: $|\tilde{X}(\pi)| = 0$

for later integration by parts in proof.

Thrm: (Time-Bandwidth Uncertainty Principle)

$$V[\tilde{x}(t)] \cdot V[\tilde{X}(w)] \geq \frac{1}{16\pi^2}$$

Prf: Since $X(w) \overset{DTFT}{=} \sum_{t=-\infty}^{\infty} x[t] e^{-iwt}$:

$$\tilde{X}'(w) = \frac{\partial}{\partial w} \tilde{X}(w) = -i \sum_{t=-\infty}^{\infty} (t \cdot \tilde{x}(t)) e^{-iwt}$$

$$\therefore \text{DTFT}(\{t \, \tilde{x}(t)\}) = i \cdot \tilde{X}'(w)$$

$$\therefore \text{Parseval: } \sum_{t=-\infty}^{\infty} t^2 \cdot |\tilde{x}(t)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{X}'(w)| \, dw$$

since $|i| = 1$

$$\therefore \sqrt{V[\tilde{x}(t)]} \cdot \sqrt{V[\tilde{X}(\omega)]}$$

$$= \sqrt{\sum_{t=-\infty}^{\infty} t^2 \cdot |\tilde{x}(t)|^2} \cdot \sqrt{\frac{1}{(2\pi)^3} \cdot \int_{-\pi}^{\pi} \omega^2 |\tilde{X}(\omega)|^2 \, d\omega}$$

Parseval

$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{X}'(\omega)|^2 \, d\omega\right)^{1/2} \left(\frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \omega^2 |\tilde{X}(\omega)|^2 d\omega\right)^{1/2}$$

Cauchy Schwartz

$$\geq \frac{1}{(2\pi)^2} \left| \int_{-\pi}^{\pi} \omega \tilde{X}^*(\omega) \tilde{X}'(\omega) \, d\omega \right|$$

$$= \frac{1}{2(2\pi)^2} \left[\left| \int_{-\pi}^{\pi} \omega \tilde{X}^*(\omega) \tilde{X}'(\omega) \, d\omega \right| + \left| \int_{-\pi}^{\pi} \omega \tilde{X}(\omega) \tilde{X}^{*\prime}(\omega) \, d\omega \right| \right]$$

$$= \frac{1}{2(2\pi)^2} \left| \int_{-\pi}^{\pi} \omega \left[\tilde{X}^*(\omega) \tilde{X}'(\omega) + \tilde{X}(\omega) \cdot \tilde{X}^{*\prime}(\omega) \right] d\omega \right|$$

$$= \frac{1}{2(2\pi)^2} \left| \int_{-\pi}^{\pi} \underbrace{\omega}_{u} \underbrace{\frac{d}{d\omega} \left(|\tilde{X}(\omega)|^2\right) d\omega}_{dv} \right| \quad \text{since } |\tilde{X}|^2 = \tilde{X}^* \tilde{X}$$

$$\int u \, dv = uv - \int v \, du \qquad u = \omega \quad du = d\omega \qquad dv = \frac{d}{d\omega} |\tilde{X}|^2 d\omega \quad v = |\tilde{X}(\omega)|^2$$

$$= \frac{1}{2(2\pi)^2} \left| \underbrace{\omega |\tilde{X}(\omega)|^2 \Big|_{-\pi}^{\pi}}_{= 0 \text{ since } |\tilde{X}(\pi)| = 0} - \underbrace{\int_{-\pi}^{\pi} |\tilde{X}(\omega)|^2 d\omega}_{= 2\pi \text{ since non-normalized pdf}} \right|$$

$$= \frac{2\pi}{2(2\pi)^2} = \frac{1}{4\pi}.$$

$$\therefore V[\tilde{x}(t)] \cdot V[\tilde{X}(\omega)] \geq \frac{1}{16\pi^2}$$

QED.